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MODELING, IDENTIFICATION AND PREDICTION OF A CLASS OF NONLINEAR VISCOELASTIC MATERIALS (II)

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Abstract—In this paper we extend the results of a previous investigation on the identification of a class of nonlinear viscoelastic materials, by generalizing the family of strain–stress histories to be employed in the identification procedure. A thoroughly detailed example is presented using the same plastic material used in the preceding investigation.

1. INTRODUCTION

IN TWO recent publications [1, 2], we have developed system identification procedures for nonlinear viscoelastic materials governed by nonlinear Volterra integral equations of the type

$$\sigma(t) = g(\varepsilon)+\int_0^t h(\varepsilon(\tau))f(t-\tau)\,\mathrm{d}\tau, \tag{1}$$

where σ and ε are scalar components of the stress and strain respectively, measured on a test specimen subject to uniaxial tension or compression. Functions f, g and h may be given in various parametric forms, in terms of a number of unknown constants. The identification problem consists of the determination of those constants by using a number of experimentally obtained strain and stress histories. In [1] we developed an identification method applicable for general input–output pairs $\sigma-\varepsilon$. In [2] we restricted the ε-inputs (or alternatively, the σ-inputs) to be given in terms of one dimensional step functions i.e. of the form

$$\varepsilon_i = \varepsilon_{i0}H(t), \tag{2}$$

where $H(t)$ is the unit step function, and where ε_{i0}, $i = 1, 2, \ldots K$, are various strain levels. This restriction leads to a simplification in the identification procedure, as might be expected. In fact, in this case we can determine function f first, independently of g and h; then, in a second stage of the computational process, we can determine the remaining functions g and h. The advantage of a decomposition of this type is obvious, but it is made at the expense of a severe limitation on the input family.

In this paper we return to consideration of general input–output (σ, ε) pairs. By using a method similar to that developed in [1] we determine, in an optimal fashion, the constants appearing in appropriate parametric expressions of functions f, g and h in equation (1), for the same polyurethane material studied in [2]. The difference here is that this time we employ generalized $(\sigma$–$\varepsilon)$ pairs to perform the identification, rather than the one dimensional ε-step functions employed in [2]. In so proceeding we pursue several objectives. In fact, in addition to furnishing an example of the method developed in [1] by using data of a real material, we wish to exhibit the advantages and disadvantages of employing general, as opposed to specific, input–output pairs, the influence of various parametric representations of function f, and other similar modeling and identification considerations.

2. FORMULATION OF THE IDENTIFICATION PROBLEM

The specific form chosen for equation (1) is given by

$$\sigma = c_1\varepsilon + c_2\varepsilon^2 + c_3 \int_0^t \varepsilon\, e^{c_4\varepsilon}(t-\tau)^{-c_5}\, d\tau, \tag{3}$$

i.e. functions f, g and h in (1) are given by

$$\begin{aligned} f(t) &= c_3 t^{-c_5} \\ g(\varepsilon) &= c_1\varepsilon + c_2\varepsilon^2, \\ h(\varepsilon) &= \varepsilon\, e^{c_4\varepsilon}, \end{aligned} \tag{4}$$

where $c = (c_j)$, $j = 1, 2, \ldots 5$, is a constant vector to be determined. To carry out the identification we are given K pairs of input–output measurements denoted by $(\bar{\varepsilon}_i, \bar{\sigma}_i)$, $i = 1, 2, \ldots K$. The determination of the unknown vector c is performed by requiring that the quadratic error functional

$$J(c) = \sum_{i=1}^{K} \eta_i \int_0^{T_i} (\bar{\sigma}_i - \sigma_i)^2\, dt, \tag{5}$$

where T_i is the duration of the ith experiment, η_i are suitable weighting factors and where σ_i is given by

$$\sigma_i(t) = c_1\bar{\varepsilon}_i(t) + c_2\bar{\varepsilon}_i^2(t) + c_3 \int_0^t \bar{\varepsilon}_i(\tau)\, e^{c_4\bar{\varepsilon}_i(\tau)}(t-\tau)^{-c_5}\, d\tau, \tag{6}$$

be a minimum with respect to every admissible vector c. When the stress outputs $\bar{\sigma}_i(t)$ are furnished at discrete values of t, equation (5) will be replaced by the discrete functional

$$J(c) = \sum_{i=1}^{K} \eta_i \sum_{k=1}^{M} (\bar{\sigma}_{ik} - \sigma_{ik})^2, \tag{7}$$

where $\bar{\sigma}_{ik} = \bar{\sigma}_i(t_k)$ and $\sigma_{ik} = \sigma_i(t_k)$.

3. SOLUTION METHOD—QUASILINEARIZATION

The solution of the optimization problem posed in the last section can be obtained by using a Gauss–Newton procedure as described in [1]. Similar equations may be obtained by quasilinearization. In fact, regarding σ_i given by (6) as a function of the unknown vector c, and quasilinearizing only with respect to c, we obtain

$$\sigma_i^{(n+1)} = \sigma_i^{(n)} + (\operatorname{grad} \sigma_i^{(n)}, \Delta c^{(n+1)}), \tag{8}$$

where (a,b) denotes the inner product of vectors a and b, $\operatorname{grad} \sigma_i^{(n)}$ stands for the gradient

$$\operatorname{grad} \sigma_i^{(n)} = \left(\frac{\partial \sigma_i^{(n)}}{\partial c_j^{(n)}} \right), \tag{9}$$

and where $\Delta c^{(n+1)}$ is the correction needed to upgrade the estimate $c^{(n)}$ at the $(n+1)$th iteration, i.e.

$$\Delta c^{(n+1)} = c^{(n+1)} - c^{(n)}. \tag{10}$$

Clearly, $\sigma_i^{(n)}$ denotes the value of σ_i in (6) evaluated at $c = c^{(n)}$. The vector $\operatorname{grad} \sigma_i^{(n)}$ can be evaluated by differentiation in equation (6), yielding

$$\operatorname{grad} \sigma_i^{(n)} = \begin{pmatrix} \bar{\varepsilon}_i(t) \\ \bar{\varepsilon}_i^2(t) \\ \displaystyle\int_0^t \bar{\varepsilon}_i(\tau)\, e^{c_4^{(n)} \bar{\varepsilon}_i(\tau)} (t-\tau)^{-c_5^{(n)}}\, d\tau \\ \displaystyle c_3^{(n)} \int_0^t \bar{\varepsilon}_i^2(\tau)\, e^{c_4^{(n)} \bar{\varepsilon}_i(\tau)} (t-\tau)^{-c_5^{(n)}}\, d\tau \\ \displaystyle -c_3^{(n)} \int_0^t \bar{\varepsilon}_i(\tau)\, e^{c_4^{(n)} \bar{\varepsilon}_i(\tau)} (t-\tau)^{-c_5^{(n)}} \ln(t-\tau)\, d\tau \end{pmatrix}. \tag{11}$$

Now, substitution $\sigma_i^{(n+1)}$ given by (8) in the equation (5) and minimization with respect to Δc^{n+1} yields

$$\sum_{i=1}^{K} \eta_i \int_0^{T_i} (\operatorname{grad} \sigma_i^{(n)}, \Delta c^{(n+1)}) \operatorname{grad} \sigma_i^{(n)}\, dt = \sum_{i=1}^{K} \eta_i \int_0^{T_i} (\bar{\sigma}_i - \sigma_i^{(n)}) \operatorname{grad} \sigma_i^{(n)}\, dt, \tag{12}$$

a linear system of algebraic equations in the five components of vector $\Delta c^{(n+1)}$, the correction of c at the $(n+1)$th iteration. This process is quadratically convergent, if convergent at all. The process starts by using a convenient initial guess for the unknown vector $c^{(0)}$. An appropriate choice of $c^{(0)}$ may considerably speed up the calculations. In general, some *a priori* knowledge of the class of material to be identified permits the construction of a suitable initial approximation. When no information at all is available, we may take $c^{(0)} \approx 0$. This is what we assume in the examples presented in this paper.

4. COMPUTATIONAL ASPECTS

If with $(\alpha_{ji}^{(n)})$, $j = 1, 2, \ldots 5$, we denote the $5 \times K$ matrix whose columns are vectors $\operatorname{grad} \sigma_i^{(n)}$, $i = 1, 2, \ldots K$, then equation (12) may be compactly written

$$A^{(n)} \Delta c^{(n+1)} = b^{(n)}, \tag{13}$$

where $A^{(n)}$ is the symmetric 5×5 matrix whose elements a_{jk} are given by

$$a_{jk}^{(n)} = a_{kj}^{(n)} = \sum_{i=1}^{K} \eta_i \int_0^{T_i} \alpha_{ji}^{(n)}\alpha_{ki}^{(n)}\,\mathrm{d}t, \qquad k,j = 1,2,\ldots 5, \tag{14}$$

and where $b^{(n)}$ is the vector whose components are given by

$$b_j^{(n)} = \sum_{i=1}^{K} \eta_i \int_0^{T_i} (\bar{\sigma}_i - \sigma_i^{(n)})\alpha_{ji}^{(n)}\,\mathrm{d}t. \tag{15}$$

The evaluation of the quantities $a_{jk}^{(n)}$ and $b_j^{(n)}$ can be conveniently reduced to the solution of an initial value differential system. The central idea involved here is to reduce the computation of the convolutions appearing in the expressions for $\sigma_i^{(n)}$ and grad $\sigma_i^{(n)}$, given by equations (6) and (11), respectively, to the solution of a system of ordinary differential equations subject to initial conditions. This can be achieved by using differential approximation to approximate the kernels of the convolutions to sums of exponentials as described in our previous paper [2], and by a subsequent reduction of the integral equations with the approximate kernels, to a system of ordinary differential equations as described in [1]. Following these lines we can compute all the quantities involved in equation (13) using no storage of auxiliary quantities. For the sake of conciseness we shall not give the details of those procedures here.

5. IDENTIFICATION OF SOLID POLYURETHANE

In [2] we have performed the identification of solid polyurethane, a polymer material experimentally investigated by Lay and Findley in [3], by using a model of the form

$$\sigma(t) = k_1\varepsilon(t) + k_2\varepsilon^2(t) + \int_0^t \varepsilon(\tau)\,\mathrm{e}^{\mathrm{d}\varepsilon(\tau)} f(t-\tau)\,\mathrm{d}\tau, \tag{16}$$

where f is a function that satisfies the Nth order differential equation

$$a_0 f + a_1\frac{\mathrm{d}f}{\mathrm{d}t} + \ldots + \frac{\mathrm{d}^N f}{\mathrm{d}f^N} = 0, \tag{17}$$

with known initial conditions. The problem consisted in the optimal determination of the unknown constants $k_1, k_2, d, a_0, a_1, \ldots a_{N-1}$, from a number of given experimental (σ, ε) histories. In that investigation, only ε-inputs such as (2) i.e. only relaxation tests at various constant strains, were allowed for identification purposes. In the present investigation we relax that requirement and, in fact, we allow for general input–output pairs. The model to be used here is that given by equation (3). We observe that both models are similar, differing only in the parametric representation of the kernel f. In equation (16) we use a differential equation such as (17) for f, while in equation (3) we employ the expression

$$f(t) = c_3 t^{-c_5}, \tag{18}$$

in terms of two unknown constants.

The following strain histories:

Strain history I

$$\varepsilon(t) = \begin{cases} 4{\cdot}333H(t)\times 10^{-3}\text{ in./in.}, & t \leq 2, \\ 6{\cdot}500H(t)\times 10^{-3}\text{ in./in.}, & t \leq 2, \\ 8{\cdot}666H(t)\times 10^{-3}\text{ in./in.}, & t \leq 2, \\ 10{\cdot}833H(t)\times 10^{-3}\text{ in./in.}, & t \leq 2, \end{cases} \tag{19}$$

Strain history II

$$\varepsilon(t) = 4{\cdot}333[H(t)+H(t-2)-H(t-3)]\times 10^{-3}\text{ in./in.}, \qquad t \leq 4, \tag{20}$$

Strain history III

$$\varepsilon(t) = [10{\cdot}833H(t)-4{\cdot}333H(t-2)]\times 10^{-3}\text{ in./in.}, \qquad t \leq 3, \tag{21}$$

whose corresponding stress histories have been experimentally determined at constant temperature and given in [3], were used for identification purposes. A computer program, based in the theory previously developed, was written such as to deal with general (ε, σ) histories. Here no use was made of the fact that the ε-histories are given by piecewise constant functions.

In order to facilitate the comparison of our present results with those obtained in [2], two separate identifications were carried out. In the first one we employed the four strain histories I, while in the second one we used concurrently the strain histories II and III. Using the two sets of five parameters determined in both identifications, we subsequently proceeded to predict the stresses associated with the strain histories I, II and III, in order to check the predictive ability of the model, relative to the information used for its identification. The resulting stresses obtained using the parameters determined with the data I are denoted by D_1 while those obtained using the parameters determined with data II and III, are denoted by D_2. In the figures, "L and F" indicates the results of Lai and Findlay given in [3].

6. NUMERICAL RESULTS

For computational purposes, the data was prepared as follows. The strain histories were entered using the analytical expressions given by (19), (20) and (21), where time is given in hours. The stresses were determined graphically, by directly measuring the corresponding ordinates in appropriately enlarged pictures of the drawings in reference [3]. The values of t for which the ordinates were measured are not equally spaced. Forty-three points in each experiment were measured in the stress histories I, seventy-one in the stress history II and sixty in the stress history III.

The values of the unknown constants $c_1, c_2 \ldots c_5$, obtained in the two identifications are given in Table 1. It is noted that both identifications have been performed using the same initial approximation and the same lower and upper permissible limits. To reach the same convergency rate, given by

$$\left|\frac{c_j^{(n+1)}-c_j^{(n)}}{c_j^{(n+1)}}\right| \leq 10^{-5}, \qquad j = 1, \ldots, 5,$$

TABLE 1.

Parameter	Initial approximation	Lower permissible limit	Upper permissible limit	Identification using (σ, ε) histories: I (17 iterations)	Identification using (σ, ε) histories: II and III (22 iterations)
$c_1[10^2$ ksi$]$	0·01	—	—	5·2358	5·1141
$c_2[10^4$ ksi$]$	0·01	—	—	−0·39263	−0·25744
$c_3[10^2$ ksi/hr$^{1-c_5}]$	−0·0099	—	—	−0·055363	−0·032759
$c_4[10^2]$	0·01	0·01	10·0	0·47245	1·0676
c_5	0·01	0·01	0·999	0·75872	0·74664

the identification employing data I needed 17 iterations, while the one employing data II and III needed 22 iterations.

TABLE 2

Stress–strain history		Mean square stress error $[10^{-5}]$ obtained using parameters identified with data: I	II and III
I	$\varepsilon = 4{\cdot}333 \times 10^{-3}$	1·19	6·36
	$\varepsilon = 6{\cdot}500 \times 10^{-3}$	5·16	6·27
	$\varepsilon = 8{\cdot}666 \times 10^{-3}$	13·36	47·37
	$\varepsilon = 10{\cdot}833 \times 10^{-3}$	7·16	18·38
II		42·49	14·20
III		38·04	13·90
Total mean square error		107·40	106·48

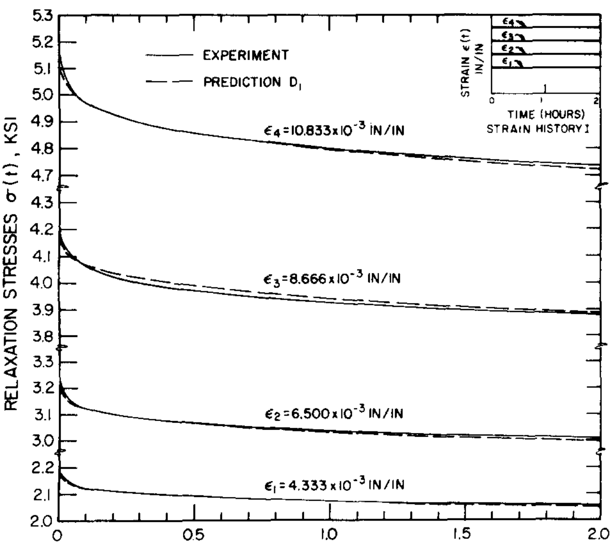

FIG. 1. Prediction of relaxation stresses under strain history I.

The predictive ability of both identifications may be measured by the mean square stress error given by $\sum_{k=1}^{R}(\sigma_{ik}^{\text{pred}}-\sigma_{ik}^{\text{exp}})^2/R$ and displayed in Table 2 on previous page. The stress histories predicted by both sets of parameters have been graphically displayed in Figs. 1–3.

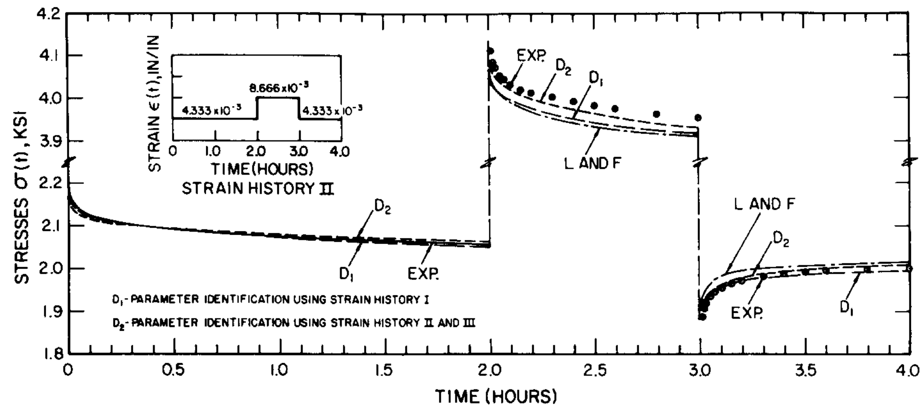

FIG. 2. Prediction of stresses under variable strain history II.

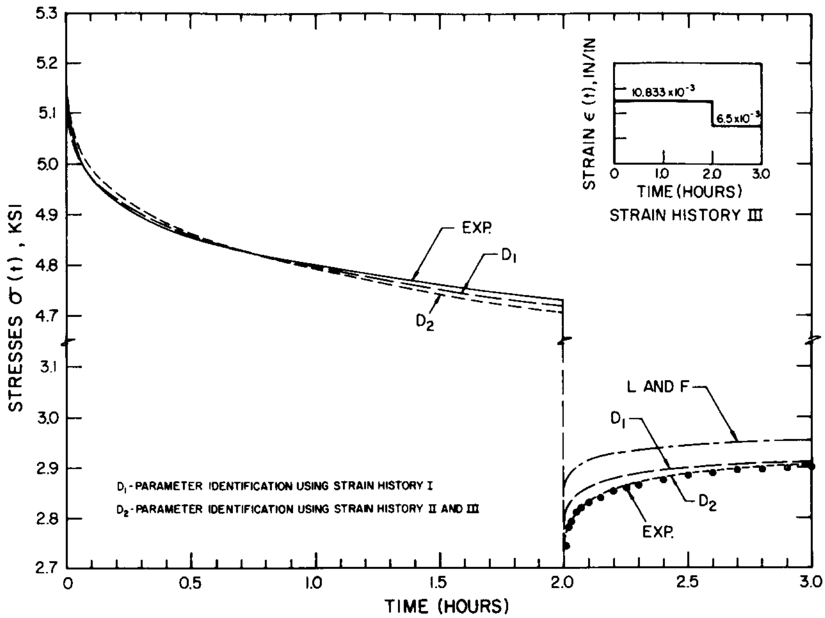

FIG. 3. Prediction of stresses under variable strain history III.

7. CONCLUDING REMARKS

We have performed the identification of solid polyurethane using various experimentally given stress–strain histories. The interest of procedures of this kind lies in its generality, since to perform the identification there is no need to provide inputs of any special class. This flexibility results in a uniform improvement of the overall predictive quality of the model. Compare for example the results displayed in Figs. 1–3 with those appearing in the corresponding figures of Ref. [2].

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Абстракт—В работе применяется нелинейное интегральное уравнение Вольтерры для изображения некоторого класса нелинейных вязкоупругих материалов, полверженных действию одноосных эффектов. На основе оптимализации, находится число неизвестных параметров, проявляющихся в математической структуре модели, которые касаются сведения до минимума функционала наименьших квадратов, из всех экспериментальных данных. Предполагается что эти данные выражаются испытаниями релаксации, для разных уровней деформации. Проверяется предсказанная способность модели, путем применения экспериментальных данных, предпоженных в/10/. Окончательно, сравниваются результаты с такими-же, сообщенными Финдлейом и Ляи, которые пользовались кратной интегральной моделью, и также с данными из экспериментов.